CHAPTER 12
LINEAR EQUATIONS IN TWO VARIABLES

Thus far in this course, discussions of equations have been limited to linear equations in one variable. Linear equations which have two variables are common, and their solution involves extending some of the procedures which have already been introduced.

RECTANGULAR COORDINATES

An outstanding characteristic of equations in two variables is their adaptability to graphical analysis. The rectangular coordinate system, which was introduced in chapter 3 of this course, is used in analyzing equations graphically. This system of vertical and horizontal lines, meeting each other at right angles and thus forming a rectangular grid, is often called the Cartesian coordinate system. It is named after the French philosopher and mathematician, Rene Descartes, who invented it.

COORDINATE AXES

The rectangular coordinate system is developed on a framework of reference similar to figure 3-2 in chapter 3 of this course. On a piece of graph paper, two lines are drawn intersecting each other at right angles, as in figure 12-1. The vertical line is usually labeled with the capital letter Y and called the Y axis. The horizontal line is usually labeled with the capital letter X and called the X axis. The point where the X and Y axes intersect is called the ORIGIN and is labeled with the letter o.

Above the origin, numbers measured along or parallel to the Y axis are positive; below the origin they are negative. To the right of the origin, numbers measured along or parallel to the X axis are positive; to the left they are negative.

COORDINATES

A point anywhere on the graph may be located by two numbers, one showing the distance of the point from the Y axis, and the other showing the distance of the point from the X axis.

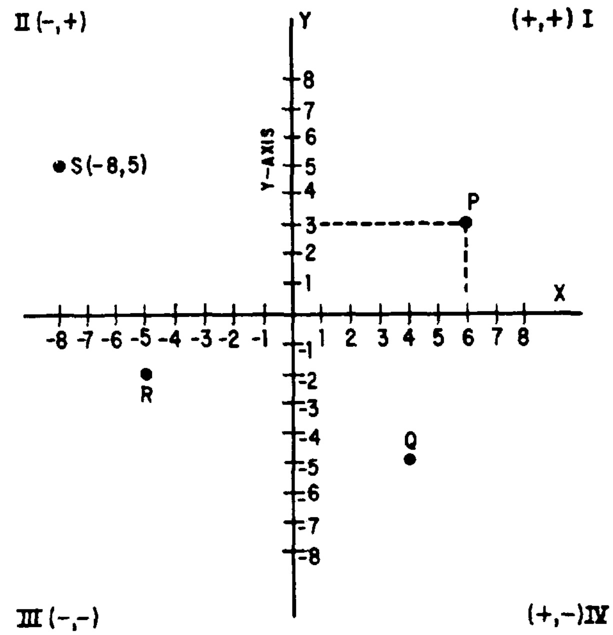

Figure 12-1.—Rectangular coordinate system.

Point P (fig. 12-1) is 6 units to the right of the Y axis and 3 units above the X axis. We call the numbers that indicate the position of a point COORDINATES. The number indicating the distance of the point measured horizontally from the origin is the X coordinate (6 in this example), and the number indicating the distance of the point measured vertically from the origin (3 in this example) is the Y coordinate.

In describing the location of a point by means of rectangular coordinates, it is customary to place the coordinates within parentheses and separate them with a comma. The X coordinate is always written first. The coordinates of point P (fig. 12-1) are written (6, 3). The coordinates for point Q are (4, -5); for point R, they are (-5, -2); and for point S, they are (-8, 5).

Usually when we indicate a point on a graph, we write a letter and the coordinates of the point. Thus, in figure 12-1, for point S, we write S(-8, 5). The other points would ordinarily be written, P(6, 3), Q(4, -5), and R(-5, -2). The Y coordinate of a point is often called its ORDINATE and the X coordinate is often called its ABSCISSA.

QUADRANTS

The X and Y axes divide the graph into four parts called QUADRANTS. In figure 12-1, point P is in quadrant I, point S is in quadrant II, R is in quadrant III, and Q is in quadrant IV. In the first and fourth quadrants, the X coordinate is positive, because it is to the right of the origin. In the second and third quadrant it is negative, because it is to the left of the origin. Likewise, the Y coordinate is positive in the first and second quadrants, being above the origin; it is negative in the third and fourth quadrants, being below the origin. Thus, we know in advance the signs of the coordinates of a point by knowing the quadrant in which the point appears. The signs of the coordinates in the four quadrants are shown in figure 12-1.

Locating points with respect to axes is called PLOTTING. As shown with point P (fig. 12-1), plotting a point is equivalent to completing a rectangle that has segments of the axes as two of its sides with lines dropped perpendicularly to the axes forming the other two sides. This is the reason for the name "rectangular coordinates."

PLOTTING A LINEAR EQUATION

A linear equation in two variables may have many solutions. For example, in solving the equation 2x - y = 5, we can find an unlimited number of values of x for which there will be a corresponding value of y. When x is 4, y is 3, since (2 x 4) - 3 = 5. When x is 3, y is 1, and when x is 6, y is 7. When we graph an equation, these pairs of values are considered coordinates of points on the graph. The graph of an equation is nothing more than a line joining the points located by the various pairs of numbers that satisfy the equation.

To picture an equation, we first find several pairs of values that satisfy the equation. For example, for the equation 2x - y = 5, we assign several values to x and solve for y. A convenient way to find values is to first solve the equation for either variable, as follows:

$$2x - y = 5$$
$$-y = -2x + 5$$
$$y = 2x - 5$$

Once this is accomplished, the value of y is readily apparent when values are substituted for x. The information derived may be recorded in a table such as table 12-1. We then lay off X and Y axes on graph paper, select some convenient unit distance for measurement along the axes, and then plot the pairs of values found for x and y as coordinates of points on the graph. Thus, we locate the pairs of values shown in table 12-1 on a graph, as shown in figure 12-2 (A).

Table 12-1.—Values of x and y in the equation 2x - y = 5.

If x = ------	-2	1	3	5	6	7	8
Then y = ---	-9	-3	1	5	7	9	11

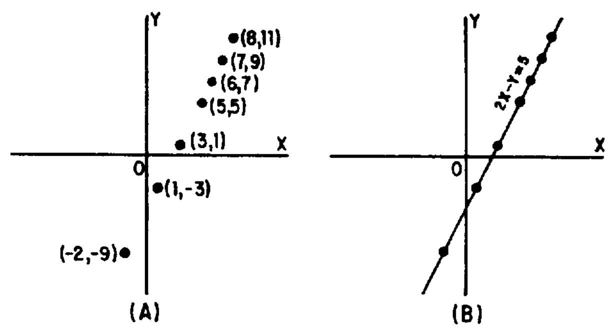

Figure 12-2.—Graph of 2x - y = 5.

Finally, we draw a line joining these points, as in figure 12-2 (B). It is seen that this is a straight line; hence the name "linear equation." Once the graph is drawn, it is customary to write the equation it represents along the line, as shown in figure 12-2 (B).

It can be shown that the graph of an equation is the geometric representation of all the points whose coordinates satisfy the conditions of the equation. The line represents an infinite number of pairs of coordinates for this equation. For example, selecting at random the point on the line where x is $2\frac{1}{2}$ and y is 0 and substituting these values in the equation, we find that they satisfy it. Thus,

$$2\left(2\frac{1}{2}\right) - 0 = 5$$

If two points that lie on a straight line can be located, the position of the line is known. The mathematical language for this is "Two points DETERMINE a straight line." We now know that the graph of a linear equation in two variables is a straight line. Since two points are sufficient to determine a straight line, a linear equation can be graphed by plotting two points and drawing a straight line through these points. Very often pairs of whole numbers which satisfy the equation can be found by inspection. Such points are easily plotted.

After the line is drawn through two points, it is well to plot a third point as a check. If this third point whose coordinates satisfy the equation lies on the line the graph is accurately drawn.

X AND Y INTERCEPTS

Any straight line which is not parallel to one of the axes has an X intercept and a Y intercept. These are the points at which the line crosses the X and Y axes. At the X intercept, the graph line is touching the X axis, and thus the Y value at that point is 0. At the Y intercept, the graph line is touching the Y axis; the X value at that point is 0.

In order to find the X intercept, we simply let $y = 0$ and find the corresponding value of x. The Y intercept is found by letting $x = 0$ and finding the corresponding value of y. For example, the line

$$5x + 3y = 15$$

crosses the Y axis at (0,5). This may be verified by letting $x = 0$ in the equation. The X intercept is (3,0), since x is 3 when y is 0. Figure 12-3 shows the line

$$5x + 3y = 15$$

graphed by means of the X and Y intercepts.

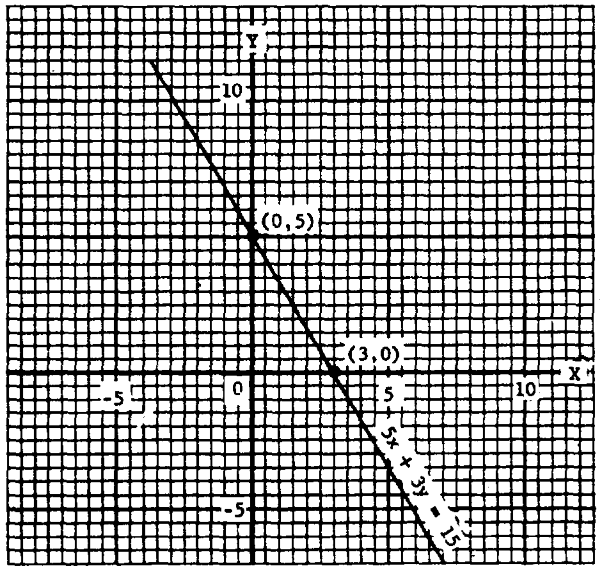

Figure 12-3.—Graph of $5x + 3y = 15$.

EQUATIONS IN ONE VARIABLE

An equation containing only one variable is easily graphed, since the line it represents lies parallel to an axis. For example, in

$$2y = 9$$

the value of y is

$$\frac{9}{2}, \text{ or } 4\frac{1}{2}$$

The line $2y = 9$ lies parallel to the X axis at a distance of $4\frac{1}{2}$ units above it. (See fig. 12-4.) Notice that each small division on the graph paper in figure 12-4 represents one-half unit.

The line $4x + 15 = 0$ lies parallel to the Y axis. The value of x is $-\frac{15}{4}$. Since this value is negative, the line lies to the left of the Y axis at a distance of $3\frac{3}{4}$ units. (See fig. 12-4.)

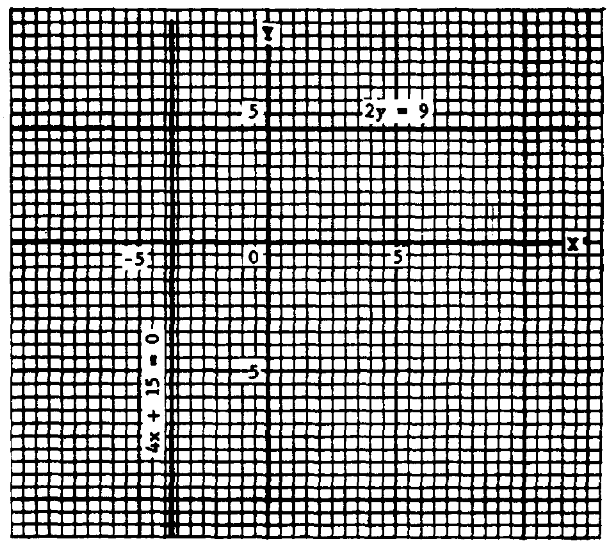

Figure 12-4.—Graphs of $2y = 9$ and $4x + 15 = 0$.

From the foregoing discussion, we arrive at two important conclusions:

1. A pair of numbers that satisfy an equation are the coordinates of a point on the graph of the equation.
2. The coordinates of any point on the graph of an equation will satisfy that equation.

SOLVING EQUATIONS IN TWO VARIABLES

A solution of a linear equation in two variables consists of a pair of numbers that satisfy the equation. For example, $x = 2$ and $y = 1$ constitute a solution of

$$3x - 5y = 1$$

When 2 is substituted for x and 1 is substituted for y, we have

$$3(2) - 5(1) = 1$$

The numbers $x = -3$ and $y = -2$ also form a solution. This is true because substituting -3 for x and -2 for y reduces the equation to an identity:

$$\begin{aligned} 3(-3) -5(-2) &= 1 \\ -9 + 10 &= 1 \\ 1 &= 1 \end{aligned}$$

Each pair of numbers (x, y) such as (2, 1) or (-3, -2) locates a point on the line $3x - 5y = 1$. Many more solutions could be found. Any two numbers that constitute a solution of the equation are the coordinates of a point on the line represented by the equation.

Suppose we were asked to solve a problem such as: Find two numbers such that their sum is 33 and their difference is 5. We could indicate the problem algebraically by letting x represent one number and y the other. Thus, the problem may be indicated by the two equations

$$x + y = 33$$
$$x - y = 5$$

Considered separately, each of these equations represents a straight line on a graph. There are many pairs of values for x and y which satisfy the first equation, and many other pairs which satisfy the second equation. Our problem is to find ONE pair of values that will satisfy BOTH equations. Such a pair of values is said to satisfy both equations at the same time, or simultaneously. Hence, two equations for which we seek a common solution are called SIMULTANEOUS EQUATIONS. The two equations, taken together, comprise a SYSTEM of equations.

Graphical Solution

If there is a pair of numbers that can be substituted for x and y in two different equations, the pair form the coordinates of a point which lies on the graph of each equation. The only way in which a point can lie on two lines simultaneously is for the point to be at the intersection of the lines. Therefore, the graphical solution of two simultaneous equations involves drawing their graphs and locating the point at which the graph lines intersect.

For example, when we graph the equations $x + y = 33$ and $x - y = 5$, as in figure 12-5, we see that they intersect in a single point. There is one pair of values comprising coordinates of that point (19, 14), and that pair of values satisfies both equations, as follows:

$x + y = 33$	$x - y = 5$
$19 + 14 = 33$	$19 - 14 = 5$

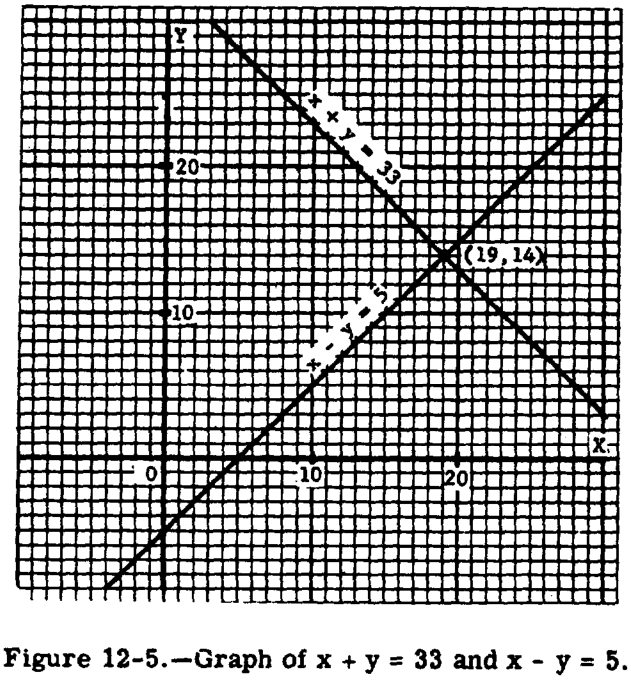

Figure 12-5.—Graph of $x + y = 33$ and $x - y = 5$.

This pair of numbers satisfies each equation. It is the only pair of numbers that satisfies the two equations simultaneously.

The graphical method is a quick and simple means of finding an approximate solution of two simultaneous equations. Each equation is graphed, and the point of intersection of the two lines is read as accurately as possible. A high degree of accuracy can be obtained but this, of course, is dependent on the precision with which the lines are graphed and the amount of accuracy possible in reading the graph. Sometimes the graphical method is quite adequate for the purpose of the problem.

Figure 12-6 shows the graphs of $x + y = 11$ and $x - y = -3$. The intersection appears to be the point (4, 7). Substituting $x = 4$ and $y = 7$ into the equations shows that this is the actual point of intersection, since this pair of numbers satisfies both equations.

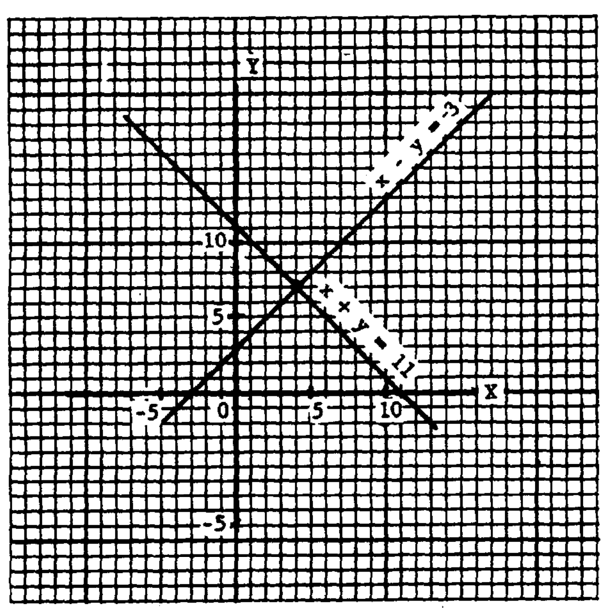

Figure 12-6.—Graph of $x + y = 11$ and $x - y = -3$.

The equations $7x - 8y = 2$ and $4x + 3y = 5$ are graphed in figure 12-7. The lines intersect where y is approximately 1/2 and x is approximately 5/6.

Practice problems. Solve the following simultaneous systems graphically:

1. $x + y = 8$; $x - y = 2$
2. $3x + 2y = 12$; $4x + 5y = 2$

Answers:

1. $x = 5$, $y = 3$
2. $x = 8$, $y = -6$

Addition Method

The addition method of solving systems of equations is illustrated in the following example:

$$\begin{aligned} x - y &= 2 \\ x + y &= 8 \\ \hline 2x + 0 &= 10 \\ x &= 5 \end{aligned}$$

The result in the foregoing example is obtained by adding the left member of the first equation to the left member of the second, and adding the right member of the first equation to the right member of the second.

Having found the value of x, we substitute this value in either of the original equations to find the value of y, as follows:

$$\begin{aligned} x - y &= 2 \\ (5) - y &= 2 \\ -y &= 2 - 5 \\ -y &= -3 \\ y &= 3 \end{aligned}$$

Notice that the primary goal in the addition method is the elimination (temporarily) of one of the variables. If the coefficient of y is the same in both equations, except for its sign, adding the equations eliminates y as in the foregoing example. On the other hand, suppose that the coefficient of the variable which we desire to eliminate is exactly the same in both equations.

In the following example, the coefficient of x is the same in both equations, including its sign:

$$\begin{aligned} x + 2y &= 4 \\ x - 3y &= -1 \end{aligned}$$

Adding the equations would not eliminate either x or y. However, if we multiply both members of the second equation by -1, then addition will eliminate x, as follows:

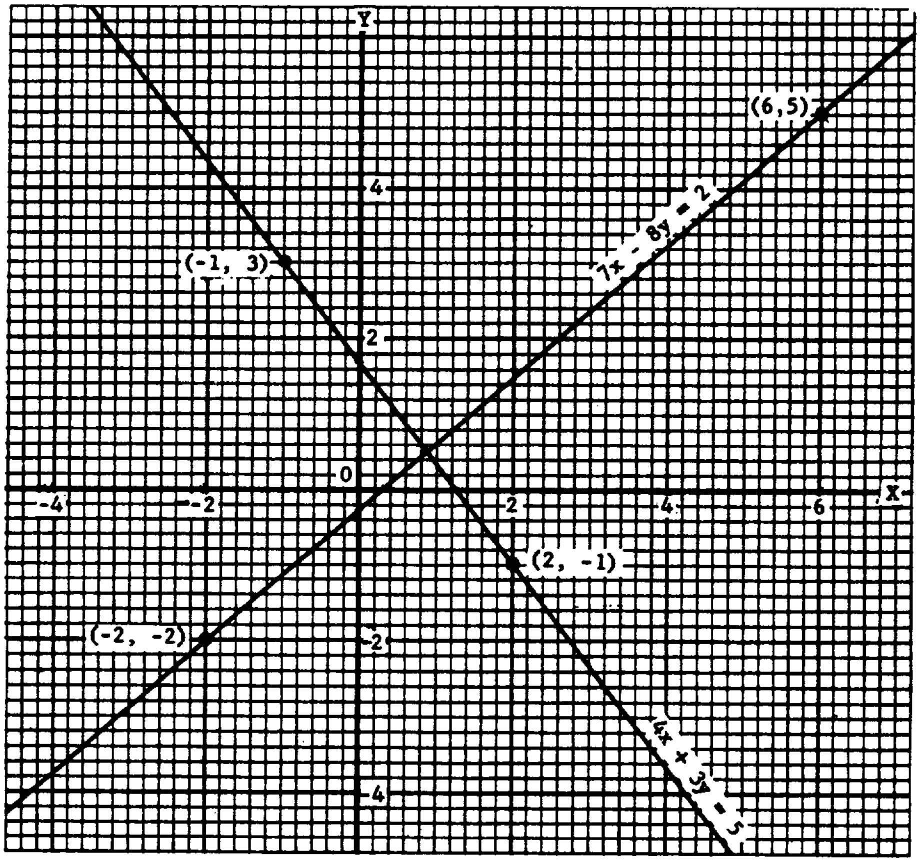

Figure 12-7.—Graph of 7x - 8y = 2 and 4x + 3y = 5.

$$\begin{array}{r} x + 2y = 4 \\ -x + 3y = 1 \\ \hline 5y = 5 \\ y = 1 \end{array}$$

The value of x is found by substituting 1 for y in either of the original equations, as follows:

$$x + 2(1) = 4$$
$$x = 2$$

As a second example of the addition method, find the solution of the simultaneous equations

$$3x + 2y = 12$$
$$4x + 5y = 2$$

Here both x and y have unlike coefficients. The coefficients of one of the variables must be made the same, except for their signs.

The coefficients of x will be the same except for signs, if both members of the first equation are multiplied by 4 and both members of the second equation by -3. Then addition will eliminate x.

Following this procedure to get the value of y, we multiply the first equation by 4 and the second equation by -3, as follows:

$$\begin{array}{r} 12x + 8y = 48 \\ -12x - 15y = -6 \\ \hline -7y = 42 \\ y = -6 \end{array}$$

Substituting for y in the first equation to get the value of x, we have

$$3x + 2(-6) = 12$$
$$x + 2(-2) = 4$$
$$x - 4 = 4$$
$$x = 8$$

This solution is checked algebraically by substituting 8 for x and -6 for y in each of the original equations, as follows:

1. $$3x + 2y = 12$$
$$3(8) + 2(-6) = 12$$
$$24 - 12 = 12$$

2. $$4x + 5y = 2$$
$$4(8) + 5(-6) = 2$$
$$32 - 30 = 2$$

Practice problems. Use the addition method to solve the following problems:

1. $x + y = 24$, $x - y = 12$
2. $5t + 2v = 9$, $3t - 2v = -5$
3. $x - 2y = -1$, $2x + 3y = 12$
4. $2x + 7y = 3$, $3x - 5y = 51$

Answers:

1. $x = 18$, $y = 6$
2. $t = 1/2$, $v = \frac{13}{4}$
3. $x = 3$, $y = 2$
4. $x = 12$, $y = -3$

Substitution Method

In some cases it is more convenient to use the substitution method of solving problems. In this method we solve one equation for one of the variables and substitute the value obtained into the other equation. This eliminates one of the variables, leaving an equation in one unknown. For example, find the solution of the following system:

$$4x + y = 11$$
$$x + 2y = 8$$

It is easy to solve for either y in the first equation or x in the second equation. Let us solve for y in the first equation. The result is

$$y = 11 - 4x$$

Since equals may be substituted for equals, we may substitute this value of y wherever y appears in the second equation. Thus,

$$x + 2(11 - 4x) = 8$$

We now have one equation that is linear in x; that is, the equation contains only the variable x.

Removing the parentheses and solving for x, we find that

$$x + 22 - 8x = 8$$
$$-7x = 8 - 22$$
$$-7x = -14$$
$$x = 2$$

To get the corresponding value of y, we substitute $x = 2$ in $y = 11 - 4x$. The result is

$$y = 11 - 4(2)$$
$$= 11 - 8$$
$$= 3$$

Thus, the solution for the two original equations is $x = 2$ and $y = 3$.

Practice problems. Solve the following systems by the substitution method:

1. $2x - 9y = 1$, $x - 4y = 1$
2. $2x + y = 0$, $2x - y = 1$
3. $5r + 2s = 23$, $4r + s = 19$
4. $t - 4v = 1$, $2t - 9v = 3$

Answers:

1. $x = 5$, $y = 1$
2. $x = 1/4$, $y = -1/2$
3. $r = 5$, $s = -1$
4. $t = -3$, $v = -1$

Literal Coefficients

Simultaneous equations with literal coefficients and literal constants may be solved for the value of the variables just as the other equations discussed in this chapter, with the exception that the solution will contain literal numbers. For example, find the solution of the system:

$$3x + 4y = a$$
$$4x + 3y = b$$

We proceed as with any other simultaneous linear equation. Using the addition method, we may proceed as follows: To eliminate the y term we multiply the first equation by 3 and the second equation by -4. The equations then become

$$\begin{aligned} 9x + 12y &= 3a \\ -16x - 12y &= -4b \\ \hline -7x \qquad &= 3a - 4b \\ x \qquad &= \frac{3a - 4b}{-7} \\ x &= \frac{4b - 3a}{7} \end{aligned}$$

To eliminate x, we multiply the first equation by 4 and the second equation by -3. The equations then become

$$\begin{aligned} 12x + 16y &= 4a \\ -12x - 9y &= -3b \\ \hline 7y &= 4a - 3b \\ y &= \frac{4a - 3b}{7} \end{aligned}$$

We may check in the same manner as that used for other equations, by substituting these values in the original equations.

INTERPRETING EQUATIONS

Recall that the general form for an equation in the first degree in one variable is $ax + b = 0$. The general form for first-degree equations in two variables is

$$ax + by + c = 0.$$

It is interesting and often useful to note what happens graphically when equations differ, in certain ways, from the general form. With this information, we know in advance certain facts concerning the equation in question.

LINES PARALLEL TO THE AXES

If in a linear equation the y term is missing, as in

$$2x - 15 = 0$$

the equation represents a line parallel to the Y axis and $7\frac{1}{2}$ units from it. Similarly, an equation such as

$$4y - 9 = 0$$

which has no x term, represents a line parallel to the X axis and $2\frac{1}{4}$ units from it. (See fig. 12-8.)

The fact that one of the two variables does not appear in an equation means that there are no limitations on the values the missing variable can assume. When a variable does not appear, it can assume any value from zero to plus or minus infinity. This can happen only if the line represented by the equation lies parallel to the axis of the missing variable.

Lines Passing Through the Origin

A linear equation, such as

$$4x + 3y = 0$$

that has no constant term, represents a line passing through the origin. This fact is obvious since $x = 0$, $y = 0$ satisfies any equation not having a constant term. (See fig. 12-8.)

Lines Parallel to Each Other

An equation such as

$$3x - 2y = 6$$

has all possible terms present. It represents a line that is not parallel to an axis and does not pass through the origin.

Equations that are exactly alike, except for the constant terms, represent parallel lines. As shown in figure 12-8, the lines represented by the equations

$$3x - 2y = -18 \text{ and } 3x - 2y = 6$$

are parallel.

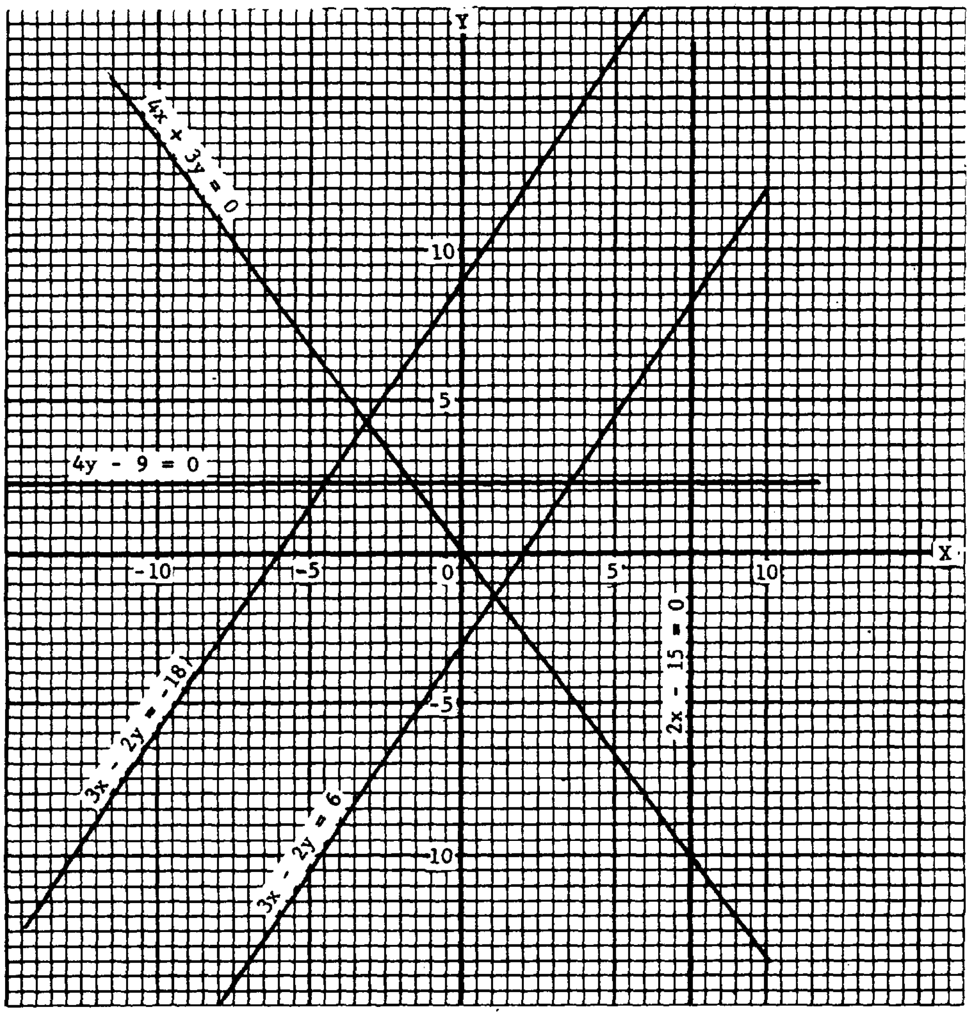

Figure 12-8.—Interpreting equations.

Parallel lines have the same slope. Changing the constant term moves a line away from or toward the origin while its various positions remain parallel to one another. Notice in figure 12-8 that the line $3x - 2y = 6$ lies closer to the origin than $3x - 2y = -18$. This is revealed at sight for any pair of lines by comparing their constant terms. That one which has the constant term of greater absolute value will lie farther from the origin. In this case $3x - 2y = -18$ will be farther from the origin since $|-18| > |6|$.

The fact that lines are parallel is indicated by the result when we try to solve two equations such as $3x - 2y = -18$ and $3x - 2y = 6$ simultaneously. Subtraction eliminates both x and y immediately. If both variables disappear, we cannot find values for them such that both equations are satisfied at the same time. This means that there is no solution. No solution implies that there is no point of intersection for the straight lines represented by the equations. Lines that do not intersect in the finite plane are parallel.

USING TWO VARIABLES IN SOLVING WORD PROBLEMS

Many problems can be solved quickly and easily using one equation with one variable. Other problems that might be rather difficult to solve in terms of one variable can easily be solved using two equations and two variables. The difference in the two methods is shown in the following example, solved first by using one variable and then using two.

EXAMPLE: Find the two numbers such that half the first equals a third of the second and twice their sum exceeds three times the second by 4.

SOLUTION USING ONE VARIABLE:

1. Let x = the first number.

2. Then $\frac{x}{2} = \frac{1}{3}$ of the second number.

3. Thus $\frac{3x}{2}$ = the second number.

From the statement of the problem, we then have

$$2\left(x + \frac{3x}{2}\right) = 3\left(\frac{3x}{2}\right) + 4$$

$$2x + 3x = \frac{9x}{2} + 4$$

$$10x = 9x + 8$$

$$x = 8 \qquad \text{(first number)}$$

$$\frac{3x}{2} = 12 \qquad \text{(second number)}$$

SOLUTION USING TWO VARIABLES:

If we let x and y be the first and second numbers, respectively, we can write two equations almost directly from the statement of the problem. Thus,

1. $\frac{x}{2} = \frac{y}{3}$

2. $2(x + y) = 3y + 4$

Solving for x in the first equation and substituting this value in the second, we have

$$x = \frac{2y}{3}$$

$$2\left(\frac{2y}{3} + y\right) = 3y + 4$$

$$\frac{4y}{3} + 2y = 3y + 4$$

$$4y + 6y = 9y + 12$$

$$y = 12 \qquad \text{(second number)}$$

$$\frac{x}{2} = \frac{12}{3}$$

$$x = 8 \qquad \text{(first number)}$$

Thus, we see that the solution using two variables is more direct and simple. Often it would require a great deal of skill to manipulate a problem so that it might be solved using one variable; whereas the solution using two variables might be very simple. The use of two variables, of course, involves the fact that the student must be able to form two equations from the information given in the problem.

Practice problems. Solve the following problems using two variables:

1. A Navy tug averages 12 miles per hour downstream and 9 miles per hour upstream. How fast is the stream flowing?

2. The sum of the ages of two boys is 18. If 4 times the younger boy's age is subtracted from 3 times the older boy's age, the difference is 12. What are the ages of the two boys?

Answers:

1. $1\frac{1}{2}$ mph.

2. 6 years and 12 years.

INEQUALITIES IN TWO VARIABLES

Inequalities in two variables are of the following form:

$$x + y > 2$$

Many solutions of such an inequation are apparent immediately. For example, x could have the value 2 and y could have the value 3, since 2 + 3 is greater than 2.

The existence of a large number of solutions suggests that a graph of the inequation would contain many points. The graph of an inequation in two unknowns is, in fact, an entire area rather than just a line.

PLOTTING ON THE COORDINATE SYSTEM

It would be extremely laborious to plot enough points at random to define an entire area of the coordinate system. Therefore our method consists of plotting a boundary line and shading the area, on one side of this line, wherein the solution points lie.

The equation of the boundary line is formed by changing the inequation to an equation. For example, the equation of the boundary line for the graph of

$$x + y > 2$$

is the equation

$$x + y = 2$$

Figure 12-9 is a graph of $x + y > 2$. Notice that the boundary line $x + y = 2$ is not solid. This is intended to indicate that points on the boundary line are not members of the solution set. Every point lying above and to the right of the boundary line is a member of the solution set. Any solution point may be verified by substituting its X and Y coordinates for x and y in the original inequation.

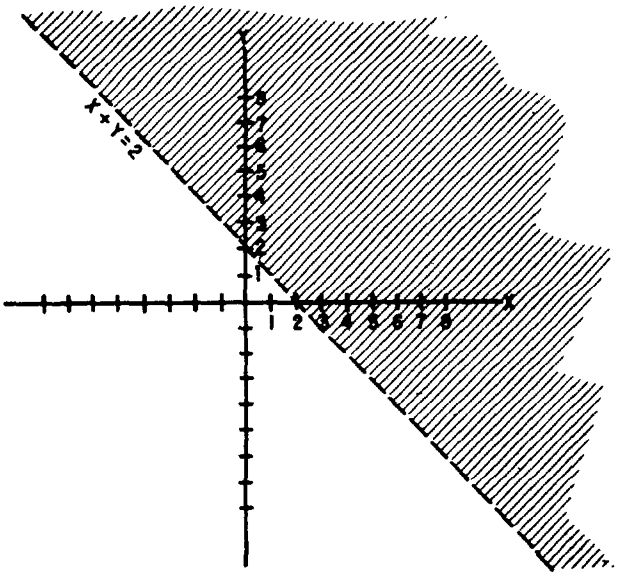

Figure 12-9.—Graph of $x + y > 2$.

SIMULTANEOUS INEQUALITIES

The areas representing the solutions of two different inequations may overlap. If such an overlap occurs, the area of the overlap includes all points whose coordinates satisfy both inequations simultaneously. An example of this is shown in figure 12-10, in which the following two inequations are graphed:

$$x + y > 2$$
$$x - y > 2$$

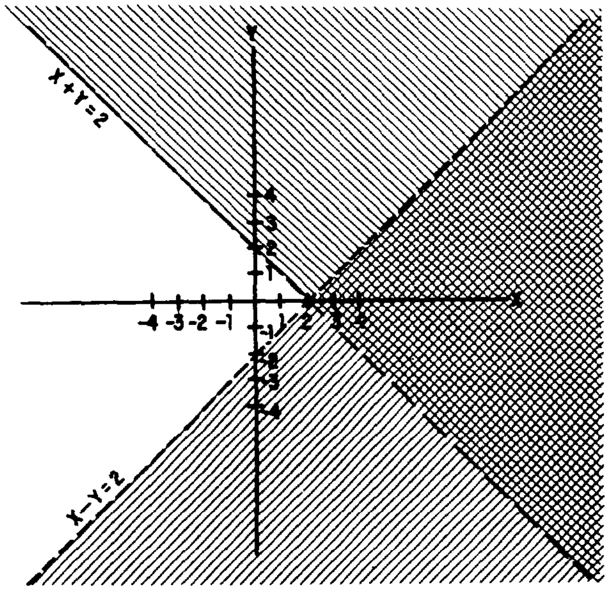

Figure 12-10.—Graph of $x + y > 2$ and $x - y > 2$.

The double crosshatched area in figure 12-10 contains all points which comprise the solution set for the system.